

Conservation laws for a class of generic Hamiltonians

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Within a strong coupling expansion, we construct local quasi-conserved operators for a class of Hamiltonians that includes both integrable and non-integrable models. We explicitly show that at the lowest orders of perturbation theory the structure of the operators is independent of the system details. Higher order contributions are investigated numerically by means of an *ab initio* method for computing the time evolution of local operators in the Heisenberg picture. The numerical analysis suggests that the quasi-conserved operators could be approximations of a quasi-local conservation law, even if the model is non-integrable.

The importance of conservation laws in physics can not be overestimated. In quantum mechanics conserved quantities are fundamental for many reasons, starting from the classification of the states (quantum numbers), the one-to-one correspondence with the symmetries of the system (Noether's theorem), the key role played in ergodicity and transport phenomena¹⁻³, the issue of integrability⁴.

In the last decade conservation laws have been in the spotlight of non-equilibrium quantum many-body physics in low dimensional systems for their influence on the relaxation of local degrees of freedom. Local conservation laws have been identified⁵⁻¹¹ as the basic elements that characterise the stationary behaviour of local observables at late time after a sudden change of a global Hamiltonian parameter (global quench). This picture was corroborated by many analytical and numerical results¹¹⁻²¹, which led to a better understanding of the type of conservation laws that are relevant for the non-equilibrium problem. Nevertheless, recent works²²⁻²⁴ have been pointing up some inconsistencies that might potentially undermine the established physical picture.

In this kind of situation, an intimate knowledge of the models under investigation is essential: disregarding a single conservation law may lead to the wrong conclusion^{11,25}.

For example, in the absence of manifest symmetries, it is widely believed that generic models do not possess (quasi-)local conservation laws. The hypothesis of local thermalisation⁵⁻⁹ is a crucial result based on that assumption. A non-integrable model could however be not as generic as one would expect²⁶⁻²⁸, resulting in behaviours difficult to understand (*cf.*²⁹).

In integrable models the situation is not clearer. By definition there is an infinite number of local conservation laws that account for integrability⁴, but other infinitely many (quasi-)local charges could be present as a result of extra symmetries^{25,30-32}. In addition, the set of conservation laws could be “oversized” and non-abelian²⁵.

In this paper we investigate a class of spin chain Hamiltonians that include many well-known integrable and non-integrable models. We carry out a strong coupling expansion and construct local operators that are quasi-

conserved.

We conjecture that the quasi-conserved operators are an approximation of a conservation law, which can be formally written without knowing the system details. To the best of our knowledge, this has never been pointed out in its full generality until now.

Finally, we investigate the locality properties of the conservation law.

a. Locality. In a spin chain we call ‘local’ a translation invariant operator \mathcal{O} that can be written as

$$\mathcal{O} = \sum_{\ell} \hat{o}_{\ell}, \quad (1)$$

where \hat{o}_{ℓ} are operators that act like the identity everywhere but on a finite number of sites around ℓ . The *range* of (1) is defined as the minimal length of the interval on which \hat{o}_{ℓ} acts non-trivially. We notice that the commutator of local operators is local.

Having in mind the non-equilibrium problem³³, we say that \mathcal{O} is ‘weakly local’ if

$$\lim_{|A| \rightarrow \infty} \|\hat{o}_{\ell} - \hat{o}_{\ell}^{(A)}\| = 0, \quad (2)$$

where $\|\cdot\|$ is the operator norm (the maximal eigenvalue in absolute value), A is a region surrounding ℓ , $|A|$ its length, and

$$\hat{o}_{\ell}^{(A)} = \frac{\text{Tr}_{\bar{A}}[\hat{o}_{\ell}]}{\text{Tr}_{\bar{A}}[\mathbf{I}_{\bar{A}}]} \otimes \mathbf{I}_{\bar{A}} \quad (3)$$

is the operator restricted to A (\bar{A} is complementary to A and $\mathbf{I}_{\bar{A}}$ is the identity on \bar{A}).

We also remind the reader of the definition of *quasi-locality* (see *e.g.*³⁴), which is a stronger form of (2) with the additional requirement of *exponential localisation* $\|\hat{o}_{\ell} - \hat{o}_{\ell}^{(A)}\| \sim e^{-|A|/r}$, where r is the *typical range*.

b. The class of models. We consider local translation invariant Hamiltonians of the form

$$H = H_0 + \Delta H_I, \quad (4)$$

where $H_0 = G + F + F^{\dagger}$ and G and F are local operators satisfying the algebra

$$[H_I, G] = 0, \quad [H_I, F] = F. \quad (5)$$

model	H_I	H_0
XYZ ⁽¹⁾	$\frac{1}{4}\{zz\}$	$J_x\{xx\} + J_y\{yy\} + \sum_{\alpha} h_{\alpha}\{\alpha\}$
XYZ ⁽²⁾	$\frac{1}{4}\{z\}$	$J_x\{xx\} + J_y\{yy\} + J_z\{zz\}$
Ising	$\frac{1}{2}\{z\}$	$J_x\{zz\} + h_x\{x\} + h_y\{y\}$
ANNNI ⁽¹⁾	$\frac{1}{4}\{x\}$	$J\{zz\} + g\{z0z\}$
ANNNI ⁽²⁾	$\frac{1}{4}\{zz\}$	$g\{z0z\} + h_x\{x\} + h_y\{y\}$
ANNNI ⁽³⁾	$\frac{1}{4}\{z0z\}$	$J\{zz\} + h_x\{x\} + h_y\{y\}$

TABLE I. A list of Hamiltonians with the property (5). They describe both integrable and non-integrable models. Notations: $\{\alpha\beta\dots\} = \sum_{\ell} \sigma_{\ell}^{\alpha} \sigma_{\ell+1}^{\beta} \dots$, where, for $\alpha \in \{x, y, z\}$, σ_{ℓ}^{α} acts like the corresponding Pauli matrix on site ℓ and like the identity elsewhere, while σ_{ℓ}^0 means identity.

This class includes widely studied models, some of which are reported in Table I (in particular, the algebraic structure (5) was used in³⁶ to compute a t/U expansion for the Hubbard model). We notice that for some models the parametrisation (4)(5) is not unique but depends on the choice of H_I .

c. Quasi-conserved operators. For large Δ we consider operators that can be series expanded in powers of Δ^{-1}

$$Q = \sum_{n=0}^{\infty} \frac{Q_n}{\Delta^n}. \quad (6)$$

We implicitly assume that the expansion makes sense. Clearly H/Δ is an operator of that kind. The commutator $[H, Q]$ can be series expanded as well

$$[H, Q] = \Delta[H_I, Q_0] + \sum_{n=0}^{\infty} \frac{[H_0, Q_n] + [H_I, Q_{n+1}]}{\Delta^n}. \quad (7)$$

We say that $Q^{(\alpha)}$ is quasi-conserved at order $O(\Delta^{-\alpha})$ if the following system is satisfied:

$$\begin{cases} [H_I, Q_0^{(\alpha)}] = 0 \\ [H_0, Q_n^{(\alpha)}] + [H_I, Q_{n+1}^{(\alpha)}] = 0 \quad n \leq \alpha. \end{cases} \quad (8)$$

In particular, we can set $Q_n^{(\alpha)} = 0$ for $n > \alpha + 1$.

The construction of conserved quantities by working out (8) is generally unfeasible and indeed in integrable models more powerful techniques have been developed⁴. We are however interested in generic models, so we can only rely on the algebra (5).

We use (5) to “integrate” H_I out of the system of equations (8). From (5) and (8) it follows that the operator $Q_n^{(\alpha)}(t) = e^{iH_I t} Q_n^{(\alpha)} e^{-iH_I t}$ is 2π -periodic in t and can therefore be expanded as a Fourier series

$$Q_n^{(\alpha)}(t) = \sum_j e^{ij t} \Omega_{n,j}, \quad (9)$$

where $\Omega_{n,-j} = \Omega_{n,j}^{\dagger}$. The problem is now reduced to the calculation of $\Omega_{n,j}$. To shorten the notations, we

introduce an auxiliary parameter s and consider a more general class of operators

$$\Omega_{n,j}(s) = e^{iGs} \Omega_{n,j} e^{-iGs}, \quad F(s) = e^{iGs} F e^{-iGs}. \quad (10)$$

Using ansatz (9), the system of equations (8) for $\Omega_{n,j}(s)$ reads as

$$\begin{cases} \dot{\Omega}_{0,0} = 0 \\ \Omega_{n,j} = 0 \quad |j| > n \\ \Omega_{n+1,j} = \frac{i\dot{\Omega}_{n,j} + [\Omega_{n,j-1}, F] + [\Omega_{n,j+1}, F^{\dagger}]}{j} \\ i\dot{\Omega}_{n,0} = [F, \Omega_{n,-1}] + [F^{\dagger}, \Omega_{n,1}], \end{cases} \quad (11)$$

where all the operators are functions of s and the dot is used to indicate the derivative with respect to s . $Q^{(\alpha)}$ is obtained going backwards through the various steps: rewriting derivatives in terms of commutators ($\dot{O} = i[G, O]$), setting $s = 0$, and summing $\Omega_{n,j}$ over j (cf. (9) with $t = 0$).

The Hamiltonian is generated by $\Omega_{0,0} = H_I$, using then (5) to remove H_I from the equations.

d. A formal conservation law. The first equation of (11) has another formal solution: $\Omega_{0,0} = G$. Let us attempt to generate quasi-conserved operators that approach G as $\Delta \rightarrow \infty$. At fixed order α , we seek a solution that can be written in terms of $F(s)$ and a finite number of its derivatives. The latter condition is sufficient for locality.

The only obstacle to achieving that goal is the inversion of the last equation of (11). If, order by order, we are able to find a primitive of $([F^{\dagger}(s), \Omega_{n,1}(s)] - \text{h.c.})$, then (11) can be solved recursively.

Before going any further, let us notice that we can always add to $\Omega_{n,0}$ a term proportional to G or H_I (cf. (11), $\dot{G} = \dot{H}_I = 0$). However, since the equations are linear in Ω , this would correspond to multiplying the full operator by a constant or to adding a term proportional to the Hamiltonian. Thus, we are going to ignore these degrees of freedom.

In Table II the solution of the system up to $O(\Delta^{-4})$ is reported. The expressions become quickly cumbersome as the order of the approximation is increased, however we stress that everything is written in terms of (nested) commutators and therefore the operators remain local.

More generally, we conjecture that $\Omega_{n,0}$ can be determined unequivocally³⁷.

The possibility of expressing the lowest orders of perturbation theory in closed form (cf. Table II) is a strong indication that $Q^{(\alpha)}$ are approximations of a conservation law Q that can be formally written without knowing the system details. From this point of view, Q is the unique formal solution of (11) independent of H for given H_I (redefining H_I other solutions can be found).

TABLE II: The lowest orders of the perturbative expansion. For $n = 4$ we only report $\Omega_{4,0}$, which is the only operator that can not be obtained with mere recursion. The shorthands in the last row stand for the nested commutators obtained by placing F , F , F^\dagger , and F^\dagger (in this exact order) between commas; the dot means derivative; *e.g.* $[[[, [, [,]]]] \equiv [F, [F, [F^\dagger, \dot{F}^\dagger]]]$.

n	$\Omega_{n,0}$	$\Omega_{n,1}$	$\Omega_{n,2}$	$\Omega_{n,3}$
0	G	0	0	0
1	$[F, F^\dagger]$	$-i\dot{F}$	0	0
2	$i([\dot{F}, F^\dagger] - [F, \dot{F}^\dagger])$	$\ddot{F} - [F, [F, F^\dagger]]$	$\frac{i}{2}[F, \dot{F}]$	0
3	$[\dot{F}, \dot{F}^\dagger] - [\ddot{F}, F^\dagger] - [F, \ddot{F}^\dagger] + \frac{3}{2}[[F, [F, F^\dagger]], F^\dagger]$	$i\ddot{F} - 3i[F, [\dot{F}, F^\dagger]] + \frac{3i}{2}[[F, \dot{F}], F^\dagger]$	$\frac{1}{2}[F, [F, [F, F^\dagger]]] + \frac{3}{4}[\ddot{F}, F]$	$\frac{i}{6}[F, [\dot{F}, F]]$
4	$i([F, \ddot{F}^\dagger] - [\dot{F}, \ddot{F}^\dagger] + [\ddot{F}, \dot{F}^\dagger] - [\ddot{F}, F^\dagger] + \frac{9}{4}[[, [, [,]]] - \frac{9}{4}[[[, [, [,]]] - 4[[, [, [,]]] + 4[[[, [, [,]]] - [, [[, [,]]] + [[, [, [,]]]]$			

Incidentally, our approach might call to mind the Schrieffer-Wolff transformation^{36,38,39}. However, it does not seem that Q can be related to the Schrieffer-Wolff effective Hamiltonian in a trivial way.

e. Locality properties. In order to gain some insights into the locality properties of Q , let us restore the dependence on Δ . The system of equations (11) is such that a factor Δ^{-1} appears for each operator F and for each derivative (*i.e.*, for each G). We therefore expect a solution of the form

$$Q = G + \Delta \mathcal{F}(\hat{f}_\Delta), \quad (12)$$

where \mathcal{F} is a functional independent of Δ and

$$\hat{f}_\Delta(s) = \frac{F(s/\Delta)}{\Delta} \quad (13)$$

is a quasi-local operator with typical range proportional to s/Δ (*cf.* (10), the proportionality constant is essentially given by the Lieb-Robinson velocity associated with G ³⁵). The larger Δ is, the smaller is the size of the region around $s = 0$ where the behaviour of $\hat{f}_\Delta(s)$ significantly affects the conservation law. In addition, because of the overall factor Δ^{-1} in (13), the contribution of nested commutators of $\hat{f}_\Delta(s)$ in \mathcal{F} is exponentially suppressed with the number of operators involved. Because each commutator increases the typical range of the operator by a finite amount, one can naively expect that Q is a quasi-local conservation law with a typical range that scales as $1/\log(\Delta/\Delta_0)$, where Δ_0 depends on the system details.

In addition, some preliminary results (see also³⁷) show that in integrable models Q is closely related to the operator obtained from the Hamiltonian by flattening the excitation energies, sending any interaction parameter different from Δ (*e.g.* g , h , and J in Table I) to zero. In non-interacting models the latter condition is sufficient for quasi-locality, however in the presence of interactions and especially in generic models the statement of quasi-locality can only be accepted with reserve.

We partially handle this weakness by studying the range of the quasi-conserved operators numerically.

f. Time averaged operators. Let us consider the time average $\bar{\mathcal{O}}$ of an operator \mathcal{O} in the Heisenberg pic-

ture

$$\bar{\mathcal{O}}(t) = \int_0^t \frac{d\tau}{t} \mathcal{O}(\tau), \quad \mathcal{O}(\tau) = e^{iH\tau} \mathcal{O} e^{-iH\tau}. \quad (14)$$

If \mathcal{O} is conserved, $\bar{\mathcal{O}}$ is independent of time and equal to \mathcal{O} . More generally, at large time the time average could approach zero, so it could be convenient to define the operator $\bar{\mathcal{O}}_*$, normalised according to a given norm

$$\bar{\mathcal{O}}_*(t) = \bar{\mathcal{O}}(t) / \|\bar{\mathcal{O}}(t)\|_* \quad (15)$$

The norm of the commutator between $\bar{\mathcal{O}}_*(t)$ and the Hamiltonian is given by

$$\|[H, \bar{\mathcal{O}}_*(t)]\|_* = \frac{\|\mathcal{O}(t) - \mathcal{O}\|_*}{t \|\bar{\mathcal{O}}(t)\|_*} \leq \frac{2 \|\mathcal{O}\|_*}{t \|\bar{\mathcal{O}}(t)\|_*}. \quad (16)$$

If $\lim_{t \rightarrow \infty} t \|\bar{\mathcal{O}}(t)\|_* = \infty$, the operator $\bar{\mathcal{O}}_*(t)$ is quasi-conserved at fixed (sufficiently large) time and conserved as $t \rightarrow \infty$. Because of the Lieb-Robinson bounds³⁵, $\bar{\mathcal{O}}_*(t)$ is a quasi-local operator. For generic local \mathcal{O} , the typical range of $\bar{\mathcal{O}}_*(t)$ could however increase in time, making $\bar{\mathcal{O}}_*(t)$ nonlocal as $t \rightarrow \infty$.

g. Numerical method. We use a simple algorithm to compute the time average of $Q^{(\alpha)}(t)$ (8) in the thermodynamic limit. It is based on the fact that commutators of translation invariant operators in spin chains can be computed efficiently. We implemented the time evolution in the Heisenberg picture by discretising the time and solving the following equation (which is exact for any $\delta t_n = t_n - t_{n-1}$)

$$\bar{\mathcal{O}}(t_n) = \left(1 - \frac{\delta t_n}{t_n}\right) e^{i\mathcal{H}\delta t_n} (\bar{\mathcal{O}}(t_{n-1})) + \frac{\delta t_n}{t_n} e^{i\mathcal{H}\delta t_n} (\mathcal{O}), \quad (17)$$

where \mathcal{H} is the superoperator associated with the time evolution $\mathcal{H}(\mathcal{O}) = [H, \mathcal{O}]$. In practice, we expanded $e^{i\mathcal{H}\delta t}$ at the second order in δt and set a lower bound (inversely proportional to the time) to the norm of the operators retained. Since the algorithm breaks unitarity, the norm of the time evolving operator is an important parameter to keep under control.

Let us focus on non-integrable models. If the quasi-conserved operators $Q^{(\alpha)}$ are approximations of a quasi-local conservation law Q , one might expect

$$\tilde{Q}^{(\alpha)}(t) \xrightarrow{t \rightarrow \infty} \gamma Q + \beta H \quad (+ \dots), \quad (18)$$

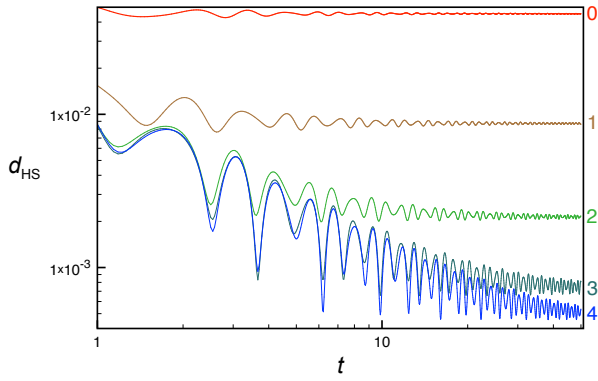


FIG. 1. The Hilbert-Schmidt distance between the (opportunistically normalised, see (24) of³⁷) time average $\tilde{G}(t)$ and the quasi-conserved operators $Q^{(\alpha)}$ for various values of α (the numbers at the end of the curves), for the quantum Ising model (19) with $g = 3/4$ and $\Delta = 6$. At large time, the larger α the smaller the distance. [Simulation parameters: The initial time step is $\delta t_0 = 0.001$ and the cutoff norm is 10^{-7} the norm of the commutator with the Hamiltonian (which is inversely proportional to the time, see the inset of Fig. 2). Both parameters are updated dynamically to partially cope with the growth of complexity]

where $\tilde{Q}^{(\alpha)}$ is the numerically computed time-average.

As a matter of fact, we can not rule out the presence of other conservation laws, also nonlocal, on the right hand side of (18). This could make us believe that Q is nonlocal even if it is not. Nevertheless, in the following we are going to assume (18).

h. Quantum Ising model. As an explicit example we consider the quantum Ising model in a magnetic field with nonzero transverse and longitudinal components

$$H = \sum_{\ell} \left(\frac{1}{4} \sigma_{\ell}^z \sigma_{\ell+1}^z + \frac{g}{2} \sigma_{\ell}^x + \frac{\Delta}{2} \sigma_{\ell}^z \right). \quad (19)$$

For generic g and Δ this describes a non-integrable model. It has however some nice properties (*e.g.* $F(s)$ is local for any s) that help reducing the numerical effort of the analysis.

Being only interested in the right hand side of (18), the quality of the numerical procedure is essentially given by how well the conservation law Q can be approximated by the time averaged quasi-conserved operators $\tilde{Q}^{(\alpha)}(t)$. In Fig. 1 the distance between the (opportunistically normalised, *cf.*³⁷) time average of G and $Q^{(\alpha)}$ is reported. Consistently with the assumption (18), the distance decreases with time (for $\alpha > 0$; notice that at time $t = 0$ the distance from $Q^{(0)} \equiv G$ is zero) and saturates to a value that is smaller, the larger α is. We now turn to the issue of locality, using the time evolution to implicitly go to higher orders of perturbation theory.

Because of its complexity, we can not use the operator norm to investigate the behaviour of the tails (2) and we must instead rely on some upper bound. An upper

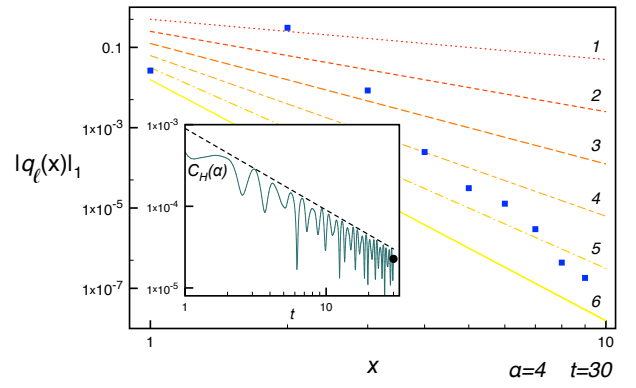


FIG. 2. The norm (20) of the operator density $q_{\ell}(x)$ (*cf.* (2)) of $\tilde{Q}_{\text{HS}}^{(4)}(t = 30)$ as a function of the range x (blue squares) for the same system as in Fig. 1. Lines are guides to the eye: they have the slopes of different power laws with the exponent displayed right above the curves. The inset can be used to estimate the relative precision, per unit of time, at which the operator is conserved: it shows $C_H(\alpha = 4) = \| [H, \tilde{Q}_{\text{HS}}^{(4)}] \|_{\text{HS}}$ as a function of time. The dashed line is inversely proportional to the time. The black circle corresponds to $t = 30$. We used the same simulation parameters as in Fig. 1.

bound that can be readily computed with our algorithm is the L^1 -norm of the coefficients of the expansion of the operator in involutions (in our case, strings of Pauli matrices):

$$\mathcal{O} = \sum_j \lambda_j S_j \Rightarrow \|\mathcal{O}\|_1 = \sum_j |\lambda_j| \quad S_j^2 = \mathbf{I}. \quad (20)$$

We notice that the inequality $\|\mathcal{O}\| \leq \|\mathcal{O}\|_1$ can be saturated when the involutions associated with nonzero coefficients commute with each other.

Figure 2 shows the norm (20) of the operators that form $\tilde{Q}^{(4)}(t)$ as a function of their range. At the time considered it is not clear whether the decay is exponential or algebraic. In the latter case, the tail could be a sign of the presence of other charges in (18). In support of this conclusion we point out that the time evolution of $Q^{(\alpha)}$ for smaller values of α results in more pronounced tails (at fixed accuracy). Nevertheless, the decay seems to be faster than a power law with an exponent sufficiently large for Q to ensure weak locality (2). In addition, a complementary analysis³⁷ in finite chains is compatible with Q being quasi-local.

Either quasi-locality or weak locality would have serious implications *e.g.* on the relaxation properties in non-integrable models; consequently, this aspect demands extreme caution. We can not exclude the existence of a timescale dependent on Δ after which $\tilde{Q}^{(\alpha)}(t)$ will display some nonlocal character. Similarly, in finite chains the quasi-local behaviour could break down at larger sizes. Nevertheless, Q is a very good candidate for a quasi-local conservation law.

i. Discussion. We have constructed local quasi-conserved operators for a class of Hamiltonians that in-

cludes both integrable and non-integrable models. Their presence is expected to leave deep marks on the time evolution of local observables, which can remain frozen in a subspace of the Hilbert space^{25,40–42}. This is particularly important in non-integrable models, in which the physical picture strongly relies on numerical studies.

We conjectured that the quasi-conserved operators are an approximation of a conservation law and investigated its locality properties. We found some numerical indications that the conservation law could be quasi-local, even in non-integrable models. Our analysis is not conclusive but suggests that pre-thermalisation or even lack of thermalisation in non-integrable models^{29,40} could be understood in terms of a small number of (quasi-)local quasi-conserved operators, independent of the system de-

tails, and possibly being approximations of weakly local (*cf.* (2)) conservation laws.

Although we focussed on a particular class of models (*cf.* (4)(5)), our construction is easily generalisable, making us wonder whether analogous conservation laws are present in any spin-chain model with a local Hamiltonian.

Due to the limitations of the perturbative approach, only the strong coupling regime was investigated. To what extent this picture survives smaller values of the coupling constant is an important question to be considered in future research.

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SUPPLEMENTAL MATERIAL

A. On the comparison of operators

The numerical construction of a conservation law is affected by ambiguities that can depend on the procedure itself and make it impossible to isolate a single charge. In the simplest situation the ambiguities can be parametrised by a few parameters. In a non-integrable model a minimal ansatz is given by (18), which we rewrite here for the sake of clarity

$$\tilde{Q}^{(\alpha)}(t) \xrightarrow{t \rightarrow \infty} \gamma Q + \beta H \quad (+ \dots), \quad (21)$$

where only the energy conservation is taken into account.

In order to compare charges, we must first enforce the equivalence $\mathcal{O} + \beta H \sim \mathcal{O}$. We have chosen to redefine the operators in such a way to minimise the Hilbert-Schmidt norm $\|\mathcal{O}\|_{\text{HS}} \propto \text{Tr}[\mathcal{O}^2]^{1/2}$. Since

$$0 = \partial_\beta \|\mathcal{O}_0 - \beta \mathcal{O}_1\|_{\text{HS}} \Rightarrow \beta = \frac{\text{tr}[\mathcal{O}_0 \mathcal{O}_1]}{\text{tr}[\mathcal{O}_1^2]}, \quad (22)$$

the minimisation results in the redefinition

$$\tilde{Q}^{(\alpha)}(t) \rightarrow \tilde{Q}^{(\alpha)}(t) - \frac{\text{Tr}[H \tilde{Q}^{(\alpha)}(t)]}{\text{Tr}[H^2]} H. \quad (23)$$

Concerning the normalisation, we prefer to keep the normalisation of $Q^{(\alpha)}$ unchanged. The normalisation of $\tilde{Q}^{(\alpha)}(t)$ can then be fixed by minimising the Hilbert-Schmidt distance between (21) and Q . Since we do not know Q , in practice we minimise the distance with a quasi-conserved operator. For example, Fig. 1 shows the time dependence of the following quantity (*cf.* (22)):

$$d_{\text{HS}}(Q^{(\alpha)}, \tilde{G}(t)) = \left\| Q^{(\alpha)} - \frac{\text{Tr}[Q^{(\alpha)} \tilde{G}(t)]}{\text{Tr}[(\tilde{G}(t))^2]} \tilde{G}(t) \right\|_{\text{HS}}, \quad (24)$$

where $Q^{(\alpha)}$ and $\tilde{G}(t)$ were redefined according to (23).

B. A non-perturbative conjecture

The formal construction of the conservation law Q relies on the non-trivial step of integrating the last equation of (11). We propose a non-perturbative conjecture that allows us to calculate $\Omega_{n,0}$ recursively. Let

$$\mathcal{K}(x) = T_\varphi e^{-ix \int_0^{2\pi} d\varphi e^{i\varphi} F + e^{-i\varphi} F^\dagger} - \sum_{n>0,j} x^n e^{ij\varphi} \Omega_{n,j}, \quad (25)$$

where T_φ is the φ -ordering operator; we claim $\mathcal{K}(x) = \text{I}$ for any x . This allows us to determine $\Omega_{n,0}$ unequivocally by series expanding $\mathcal{K}(x)$ about $x = 0$ and imposing that the (operator) coefficients are zero (except for the trivial zeroth order). The meaning and the consequences of $\mathcal{K}(x) = \text{I}$ will be explored elsewhere. We just point out that, in most of the cases, this is equivalent to say that the spectrum of $H - Q$ is equally spaced.

C. Exact diagonalisation

Working out (25) order by order can be rather involved, however the problem can in fact be overcome in finite chains.

The first step is to construct a quasi-conserved approximation of the conserved operator. This can be done *e.g.* using the perturbative results shown in Table II.

The second step is to elevate the operator to an exactly conserved quantity. To that aim we project the quasi-conserved operator on the stationary states

$$Q^{(\alpha)} \rightarrow \sum_n \langle n | Q^{(\alpha)} | n \rangle | n \rangle \langle n |, \quad (26)$$

where the sum is over an eigenbasis of the Hamiltonian and we assumed that Q does not resolve the (exact) degeneracies of H . Although the final operator is exactly conserved, it is still only an approximation of Q , being the eigenvalues only approximately correct.

The next stage is genuinely non-perturbative: we correct the eigenvalues according to (25):

$$\langle n | Q^{(\alpha)} | n \rangle = Q_n + \delta Q^{(\alpha)} \xrightarrow{(25)} Q_n. \quad (27)$$

This results in a conservation law that satisfies all our hypotheses: $[Q, H] = 0$, Q is approximated by $Q^{(\alpha)}$ at order $\mathcal{O}(\Delta^{-\alpha})$ and fulfils (25).

Having constructed Q , we can study the norm of the operators it consists of as a function of their range. In a finite chain the range can be defined as the minimal length of the connected subsystem in which the operator acts nontrivially. The operators with fixed range x smaller than $L/2 + 1$ can be singled out as follows:

$$q_\ell(x) = \frac{\text{Tr}_{\bar{A}}[Q]}{\text{Tr}[\text{I}_{\bar{A}}]} \otimes \text{I}_{\bar{A}} - \frac{\text{Tr}_{\bar{A}\cup\mathcal{R}}[Q]}{\text{Tr}[\text{I}_{\bar{A}\cup\mathcal{R}}]} \otimes \text{I}_{\bar{A}\cup\mathcal{R}} \\ - \frac{\text{Tr}_{\mathcal{L}\cup\bar{A}}[Q]}{\text{Tr}[\text{I}_{\mathcal{L}\cup\bar{A}}]} \otimes \text{I}_{\mathcal{L}\cup\bar{A}} + \frac{\text{Tr}_{\mathcal{L}\cup\bar{A}\cup\mathcal{R}}[Q]}{\text{Tr}[\text{I}_{\mathcal{L}\cup\bar{A}\cup\mathcal{R}}]} \otimes \text{I}_{\mathcal{L}\cup\bar{A}\cup\mathcal{R}}, \quad (28)$$

where $|A| = x$, ℓ denotes the position of A , and we indicated with \mathcal{R} and \mathcal{L} the site at the right and at the left edge of A , respectively. The construction is a bit more involved when the range is larger than half a chain, including (28) contributions already considered at smaller ranges, and being (28) insensitive to the symmetries of the operators.

The relation between the range in the finite chain and the range in the thermodynamic limit is clearly ambiguous when the range is comparable with L ; however, if the typical length of the operator is sufficiently smaller than the system size, the range in the finite chain is a good approximation of the one in the thermodynamic limit.

A first check of weak locality (or *pseudo-locality*³¹) is the behaviour of the operator norm of Q per unit of length as a function of the length. For $\Delta = 2$ the asymptotic value is reached so quickly that the relative difference between $\|Q\|/L$ from $L = 4$ to $L = 12$ is smaller than 10^{-8} .

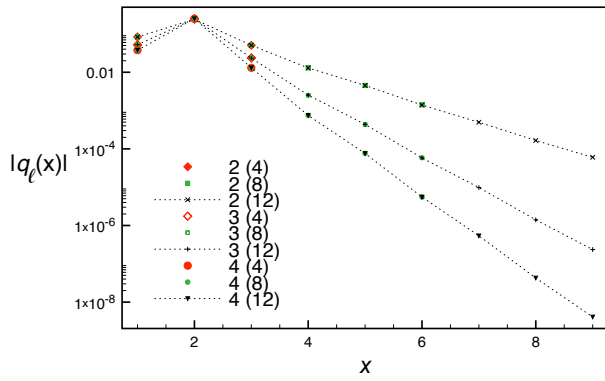


FIG. 3. The operator norm of the operator density $q_\ell(x)$ of Q as a function of the range x for the same system as in Fig. 1 and various values of Δ and system sizes (in the legend $\Delta (L)$). The tail seems to decay exponentially.

The same fast convergence is found in $\|q_\ell(x)\|$ as a function of the range x . Fig. 3 shows $\|q_\ell(x)\|$ for various values of Δ and system sizes. Our preliminary analysis in small chains is compatible with Q being quasi-local. It should not be difficult to consider larger sizes, however the construction of Q can become computationally demanding, also because, as the size is increased, the non-perturbative step requires a better quasi-conserved approximation of the conservation law.

Although this approach allows us to deal with the actual conservation law, it has an important weakness related to the order of limits. In the finite chain, because of the perturbative step, Δ turns out to be always comparable with L . On the other hand, we are interested in the limit $\Delta \ll L$, which is not easily accessible by exact diagonalisation techniques.